TOPOLOGY - III, EXERCISE SHEET 12

Exercise 1. *Homology of Grassmanians.*
We define the Grassmanian of $k-$planes in $\mathbb{C}^n$ as:

$$Gr(k,n) := \{W \subseteq \mathbb{C}^n \mid W \text{ is a linear subspace of } \mathbb{C}^n \text{ and } \dim W = k\}.$$

Let $V_k(\mathbb{C}^n) := \{(v_1,..,v_k) \in (\mathbb{C}^n)^k \mid v_1,..,v_k \text{ are linearly independent.}\}$. One can show that for all k, the sets $V_k(\mathbb{C}^n)$ are open subsets of $(\mathbb{C}^n)^k$. Observe that we have a natural surjective map

$$q : V_k(\mathbb{C}^n) \to Gr(k,n),\ (v_1,...,v_k) \mapsto \text{Span}\{v_1,...,v_k\}.$$

Therefore using q we can endow $Gr(k,n)$ with the quotient topology. Note that in particular we have $Gr(1,n+1) = \mathbb{CP}^n$. In this way Grassmanians generalise the notion of projective space and have proved to be interesting and important spaces present in many fields of mathematics such as algebraic geometry, combinatorics, representation theory and topology.

It is a fact that endowed with the above topology, $Gr(k,n)$ has a natural structure of a compact path connected complex/real/topological-manifold, with real dimension $2k(n-k)$. The goal of this exercise is to realise $Gr(k,n)$ as a CW-complex and compute its homology groups.

(1) A Schubert symbol σ is a $k-$tuple of natural numbers $(\sigma_1,...\sigma_k)$ such that $1 \leq \sigma_1 < \sigma_2 < ... < \sigma_k \leq n$. Given a Schubert symbol σ, we now define a subspace of $Gr(k,n)$ called the Schubert cell $e(\sigma)$. To this end consider the standard basis $(e_1,...,e_n)$ of $\mathbb{C}^n$ and for $k = 0,...,n$, define F_k as the span of $(e_1,...,e_k)$. Then we define the Schubert cell of σ to be

$$e(\sigma) := \{W \in Gr(k,n) \mid dim(W \cap F_{\sigma_i}) = i \text{ and } dim(W \cap F_{\sigma_i - 1}) = i-1.\}$$

Show that given $W \in Gr(k,n)$, there exists a unique Schubert symbol σ such that $W \in e(\sigma)$.

(2) Let $o_k(\mathbb{C}^n) := \{(v_1,..,v_k) \in (\mathbb{C}^n)^k \mid v_1,..,v_k \text{ are orthonormal vectors.}\} \subseteq V_k(\mathbb{C}^n)$. Also for $0 < \alpha \leq n$ let $P_\alpha := \{(b_1,...,b_\alpha,0...,0) \in \mathbb{C}^n \mid b_\alpha \in \mathbb{R}_{>0}\}$ and $\overline{P}_\alpha := \{(b_1,...,b_\alpha,0...,0) \in \mathbb{C}^n \mid b_\alpha \in \mathbb{R}_{\geq 0}\}$.

(a) Define $e'(\sigma) := o_k(\mathbb{C}^n) \cap \Pi_{i=1}^k P_{\sigma_i}$. Show that q induces a homeomorphism of $e'(\sigma)$ onto $e(\sigma)$.

Hint: Note that $\Pi_{i=1}^{k} P_{\sigma_i} \subseteq V_k(\mathbb{C}^n)$ and that given $W \in e(\sigma)$, there exists $(v_1, ..., v_k) \in \Pi_{i=1}^{k} P_{\sigma_i}$ such that $q(v_1, ..., v_k) = W$. Then show that there is a unique choice of such a tuple $(v_1, ..., v_k) \in \Pi_{i=1}^{k} P_{\sigma_i}$ such that the v_i are an orthonormal basis of W. To do this try to perform row-reduction on a $k \times n$ matrix with the $i-$ th row as v_i to obtain an orthogonal basis of W, then normalise.

(b) Show that $e'(\sigma)$ is homeomorphic to an open disc of dimension $2\sum_{i=1}^{k}(\sigma_i - i)$. Furthermore, let $\overline{e'}(\sigma) := o_k(\mathbb{C}^n) \cap \Pi_{i=1}^{k} \overline{P}_{\sigma_i}$. Show that $\overline{e'}(\sigma)$ is homeomorphic to a closed disc of dimension $2\sum_{i=1}^{k}(\sigma_i - i)$ with interior equal to $e(\sigma)$.
Hint: Consider the graph of the function $w \mapsto \sqrt{1 - |w|^2}$, with $w \in \mathbb{C}^n$ and $|w| \leq 1$.

(c) Show that if $V \in q(\overline{e'}(\sigma) \backslash e'(\sigma))$, then $V \in e(\tau)$, where the dimension of $e(\tau)$ is less than the dimension of $e(\sigma)$.

(3) Conclude from (2) that the Schubert cells $e(\sigma)$ define a CW-complex structure of $Gr(k, n)$ and the characteristic map of $e(\sigma)$ is the restriction of q to $\overline{e'}(\sigma)$.

(4) Since the dimension of $e(\sigma)$ is always even, note that all the boundary maps of the cellular complex of $Gr(k, n)$. Using this compute all the homology groups of
(a) Gr(1,4).
(b) Gr(3,4).
(c) Gr(2,4).
(d) Gr(3,5).

Exercise 2. *Cellular Boundary Formula.*
Recall that the cellular complex is a chain complex

$$... \to H_{n+1}(X_{n+1}, X_n) \xrightarrow{d_{n+1}} H_n(X_n, X_{n-1})) \xrightarrow{d_n} H_{n-1}(X_{n-1}, X_{n-2}) \to ...$$

The boundary maps d_n are defined by the composition of the natural maps $H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$.

Recall that $H_n(X_n, X_{n-1})$ can be identified as the free abelian group generated by the $n-$cells of X. With this identification, show that $d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$, where $d_{\alpha\beta}$ is the degree of the composition $S_\alpha^{n-1} \to X_{n-1} \to S_\beta^{n-1}$. Here the first map is given by attaching map of e_α^n and the second map is the quotient map obtained by contracting $X_{n-1} - e_\beta^{n-1}$ to a point.